Angles sum and difference formula:-

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A-B) + \cos(A+B) = 2\cos A \cos B$$

$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$$

$$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$$

$$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\cot(A+B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}$$

$$\cot(A-B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}$$

$$\tan\left(\frac{\pi}{4}+\theta\right) = \frac{1+\tan\theta}{1-\tan\theta}$$

$$\tan\left(\frac{\pi}{4}-\theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$$

$$\cot\left(\frac{\pi}{4}+\theta\right) = \frac{\cot\theta - 1}{\cot\theta + 1}$$

$$\cot\left(\frac{\pi}{4}-\theta\right) = \frac{\cot\theta + 1}{\cot\theta - 1}$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

☆ if $A+B+C = n\pi$

$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\cot(A+B+C) = \frac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1 - \cot A \cot B - \cot B \cot C - \cot C \cot A}$$

☆ if $A+B+C = n\pi$

$\Rightarrow \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

Two golden formulas:-

$$\sin(A+B)\ \sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$

$$\cos(A+B)\ \cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

C - D Formula:-

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \cdot \sin \frac{C-D}{2}$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$\cos C - \cos D = 2 \sin \frac{D+C}{2} \cdot \sin \frac{D-C}{2}$$

2A , 3A and $\frac{A}{2}$ angle Formula:-

$$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1+\tan^2 A}$$

$$\Rightarrow \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} = \frac{2 \tan A/2}{1+\tan^2 A/2}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\Rightarrow \cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = 2\cos^2 \frac{A}{2} - 1 = 1 - 2\sin^2 \frac{A}{2} = \frac{1-\tan^2 A/2}{1+\tan^2 A/2}$$

very useful:-

$$\cos A = \pm \sqrt{\frac{1+\cos 2A}{2}}$$

$$\sin A = \pm \sqrt{\frac{1-\cos 2A}{2}}$$

$$\tan A = \pm \sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} \Rightarrow \tan A = \frac{2\tan A/2}{1-\tan^2 A/2}$$

$$\cot 2A = \frac{\cot^2 A - 1}{2\cot A} \Rightarrow \cot A = \frac{\cot^2 A/2 - 1}{2\cot A/2}$$

$$\star \quad \tan n\theta = \frac{{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + {}^nC_5 \tan^5\theta - {}^nC_7 \tan^7\theta + \cdots}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - {}^nC_6 \tan^6\theta + \cdots}$$

$$\star \quad \cot n\theta = \frac{\cot^n\theta - {}^nC_2 \cot^{n-2}\theta + {}^nC_4 \cot^{n-4}\theta - {}^nC_6 \cot^{n-6}\theta + \cdots}{{}^nC_1 \cot^{n-1}\theta - {}^nC_3 \cot^{n-3}\theta + {}^nC_5 \cot^{n-5}\theta - {}^nC_7 \cot^{n-7}\theta + \cdots}$$

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\cot 3A = \frac{3\cot A - \cot^3 A}{1 - 3\cot^2 A}$$

$$\sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

$$\cos^3 A = \frac{3\cos A + \cos 3A}{4}$$

Some Special Formula :-

$$\sin n\theta = 2^{n-1} \sin\theta \cdot \sin\left(\frac{\pi}{n}+\theta\right) \cdot \sin\left(\frac{2\pi}{n}+\theta\right) \cdot \sin\left(\frac{3\pi}{n}+\theta\right) \cdots$$

$$\Rightarrow \sin\theta \, \sin\left(\frac{\pi}{3}+\theta\right) \sin\left(\frac{2\pi}{3}+\theta\right) = \frac{1}{4} \sin 3\theta$$

$$\Rightarrow \sin\theta \; \sin(60^\circ+\theta) \; \sin(60^\circ-\theta) = \frac{1}{4} \sin 3\theta$$

$$\cos\theta \; \cos(60^\circ+\theta) \; \cos(60^\circ-\theta) = \frac{1}{4} \cos 3\theta$$

$$\tan\theta \; \tan(60^\circ+\theta) \; \tan(60^\circ-\theta) = \tan 3\theta$$

Sum of trignometric Series :-

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin[\alpha+(n-1)\beta] = \frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)}{\sin \beta/2} \sin\frac{n\beta}{2}$$

★ if $\beta = \alpha$, then -

$$\sin\alpha + \sin 2\alpha + \sin 3\alpha + \cdots + \sin n\alpha = \frac{\sin\left(\frac{n+1}{2}\alpha\right)}{\sin \alpha/2} \sin\frac{n\alpha}{2}$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos[\alpha+(n-1)\beta] = \frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)}{\sin \beta/2} \sin n\beta/2$$

★ if $\beta = \alpha$, then -

$$\cos\alpha + \cos 2\alpha + \cos 3\alpha + \cdots + \cos n\alpha = \frac{\cos\left(\frac{n+1}{2}\alpha\right)}{\sin\frac{\alpha}{2}} \cdot \sin\frac{n\alpha}{2}$$

Multiple of angles :-

$$\cos A \; \cos 2A \; \cos 4A \; \cos 8A \cdots \cos 2^{n-1}A = \prod_{n=1}^{n} \cos 2^{n-1}\theta$$

$$= \frac{1}{2^n \sin A} \sin 2^n A$$

Value of special angles:-

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ$$

$$\sin 54^\circ = \frac{\sqrt{5}+1}{4} = \cos 36^\circ$$

यदि $0^\circ < A < 90^\circ$ तो -

$$\cos\frac{A}{2} + \sin\frac{A}{2} = \sqrt{1+\sin A}$$

$$\cos\frac{A}{2} - \sin\frac{A}{2} = \sqrt{1-\sin A}$$

Trignometric Inequallities:-

$$\boxed{\begin{array}{c} -1 \le \sin\theta \le 1 \\ -1 \le \cos\theta \le 1 \\ -\infty < \tan\theta < \infty \\ -\infty < \cot\theta < \infty \\ \sec\theta \le -1 \quad \text{and} \quad 1 \le \sec\theta \\ \operatorname{cosec}\theta \le -1 \quad \text{and} \quad 1 \le \operatorname{cosec}\theta \end{array}}$$

$0 \le \sin^2\theta \le 1$

$0 \le \cos^2\theta \le 1$

$0 \le \tan^2\theta < \infty$

$0 \le \cot^2\theta < \infty$

$1 \le \sec^2\theta < \infty$

$1 \le \operatorname{cosec}^2\theta < \infty$

$$0 \le \cdots\cdots\cdots \sin^8\theta \le \sin^6\theta \le \sin^4\theta \le \sin^2\theta \le 1$$

$$0 \le \cdots\cdots\cdots \cos^8\theta \le \cos^6\theta \le \cos^4\theta \le \cos^2\theta \le 1$$

$$1 \le \sec^2\theta \le \sec^4\theta \le \sec^6\theta \le \sec^8\theta \cdots\cdots\cdots < \infty$$

$$1 \le \operatorname{cosec}^2\theta \le \operatorname{cosec}^4\theta \le \operatorname{cosec}^6\theta \le \operatorname{cosec}^8\theta \cdots\cdots\cdots < \infty$$

$$\frac{3}{4} \le \sin^4\theta + \cos^2\theta \le 1$$

$$\frac{3}{4} \le \cos^4\theta + \sin^2\theta \le 1$$

$$\sin^2\theta + \operatorname{cosec}^2\theta \ge 2$$

$\because$ A.M. $\ge$ G.M. $\ge$ H.M.

$$\cos^2\theta + \sec^2\theta \ge 2$$

$$\tan^2\theta + \cot^2\theta \ge 2$$

$$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$$

Conditional Identity:—

if $A+B+C = \pi$

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$$

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$$

$$\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$$

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

General Solutions of Trignometric Equations:-

Equation	main value of θ	General Value of θ
$\sin\theta = 0$	$\alpha = 0$	$\theta = n\pi$; $n \in I$
$\cos\theta = 0$	$\alpha = \frac{\pi}{2}$	$\theta = (2n-1)\frac{\pi}{2}$; $n \in I$ या $\theta = (2n+1)\frac{\pi}{2}$; $n \in I$
$\tan\theta = 0$	$\alpha = 0$	$\theta = n\pi$; $n \in I$
$\sin\theta = a$	$\alpha = \sin^{-1}a$	$\theta = n\pi + (-1)^n\alpha$; $n \in I$
$\sin\theta = 1$	$\alpha = \frac{\pi}{2}$	$\theta = 2n\pi + \frac{\pi}{2}$; $n \in I$ या $\theta = n\pi + (-1)^n\frac{\pi}{2}$; $n \in I$
$\sin\theta = -1$	$\alpha = -\frac{\pi}{2}$	$\theta = 2n\pi - \frac{\pi}{2}$; $n \in I$ या $\theta = n\pi - (-1)^n\frac{\pi}{2}$
$\cos\theta = a$	$\alpha = \cos^{-1}a$	$\theta = 2n\pi \pm \alpha$; $n \in I$
$\cos\theta = 1$	$\alpha = 0$	$\theta = 2n\pi$; $n \in I$
$\cos\theta = -1$	$\alpha = \pi$	$\theta = 2n\pi \pm \pi$; $n \in I$ या $\theta = (2n+1)\pi$; $n \in I$ या $\theta = (2n-1)\pi$; $n \in I$ या $\theta = (2n\pm1)\pi$; $n \in I$

Equation	main value of θ	General Value of θ
$\tan\theta = a$	$\alpha = \tan^{-1}a$	$\theta = n\pi + \alpha$; $n \in I$
$\tan\theta = 1$	$\alpha = \frac{\pi}{4}$	$\theta = n\pi + \frac{\pi}{4}$; $n \in I$
$\tan\theta = -1$	$\alpha = -\frac{\pi}{4}$	$\theta = n\pi - \frac{\pi}{4}$; $n \in I$
$\sin^2\theta = a^2$	$\alpha = \sin^{-1}a$	$\theta = n\pi \pm \alpha$; $n \in I$
$\sin^2\theta = 1$	$\alpha = \frac{\pi}{2}$	$\theta = n\pi \pm \frac{\pi}{2}$; $n \in I$ या $\theta = (2n \pm 1)\frac{\pi}{2}$; $n \in I$ या $\theta = (2n+1)\frac{\pi}{2}$; $n \in I$ या $\theta = (2n-1)\frac{\pi}{2}$; $n \in I$
$\cos^2\theta = a^2$	$\alpha = \cos^{-1}a$	$\theta = n\pi \pm \alpha$; $n \in I$
$\cos^2\theta = 1$	$\alpha = 0$ या $\alpha = \pi$	$\theta = n\pi$; $n \in I$ या $\theta = n\pi \pm \pi$; $n \in I$
$\tan^2\theta = a^2$	$\alpha = \tan^{-1}a$	$\theta = n\pi \pm \alpha$; $n \in I$
$\tan^2\theta = 1$	$\alpha = \frac{\pi}{4}$	$\theta = n\pi \pm \frac{\pi}{4}$; $n \in I$

Properties of Triangles :—

Sine Formula (ज्या सूत्र) :-

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{abc}{2\Delta}$$

$$\Rightarrow \Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$$

where :- $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Cosine Formula (कोज्या सूत्र) :-

$$\cos A = \frac{-a^2+b^2+c^2}{2bc} \Rightarrow a^2 = b^2+c^2-2bc\cos A$$

$$\cos B = \frac{a^2-b^2+c^2}{2ac} \Rightarrow b^2 = c^2+a^2-2ca\cos B$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \Rightarrow c^2 = a^2+b^2-2ab\cos C$$

Tangent Formula (स्पर्शज्या सूत्र) :- (नेपोलियन सूत्र) (Nepier Analog)

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot\frac{C}{2}$$

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c} \cot\frac{A}{2}$$

$$\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a} \cot\frac{B}{2}$$

प्रक्षेप सूत्र :-

$$a = b\cos C + c\cos B$$
$$b = c\cos A + a\cos C$$
$$c = a\cos B + b\cos A$$

sine Formula for Half Angle :-
(अर्द्ध कोणों के लिए ज्या सूत्र)

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

Cosin Formula for Half Angles :-
(अर्द्ध कोणों के लिए कोज्या सूत्र)

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

Tangent Formula for Half Angles :-
(अर्द्ध कोणों के लिए स्पर्शज्या सूत्र)

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{\Delta}{s(s-b)}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{\Delta}{s(s-c)}$$

$$\Delta = s(s-a)\tan\frac{A}{2} = s(s-b)\tan\frac{B}{2} = s(s-c)\tan\frac{C}{2}$$

☆ $\cos A + \cos B + \cos C$ का Max. मान $= \frac{3}{2}$
if A, B, C are angles of ΔABC.

Properties of Triangle Related to Circle :-

त्रिभुज की परित्रिज्या R के साथ:-

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{abc}{2\Delta}$$

$$\Rightarrow \quad a = 2R\sin A$$

$$b = 2R\sin B$$

$$c = 2R\sin C$$

$$R = \frac{abc}{4\Delta}$$

$$R = \frac{abc}{4rs} = \frac{abc}{4r_1(s-a)} = \frac{abc}{4r_2(s-b)} = \frac{abc}{4r_3(s-c)}$$

त्रिभुज की अंतः त्रिज्या r के साथ:-

$$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = \frac{\Delta}{s}$$

$$r = \frac{a\sin\frac{C}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$$

त्रिभुज की बहिःत्रिज्याओं r_1, r_2, r_3 के साथ:-

$$\boxed{\begin{aligned} r_1 &= s\tan\frac{A}{2} \\ r_2 &= s\tan\frac{B}{2} \\ r_3 &= s\tan\frac{C}{2} \end{aligned}}$$

$$\boxed{\begin{aligned} r_1 &= \frac{\Delta}{(s-a)} \\ r_2 &= \frac{\Delta}{(s-b)} \\ r_3 &= \frac{\Delta}{(s-c)} \end{aligned}}$$

$$\boxed{\begin{aligned} r_1 &= \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} \\ r_2 &= \frac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}} \\ r_3 &= \frac{b\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}} \end{aligned}}$$

R, r में संबंध:—

$$R = \frac{abc}{4rs} \qquad r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$$

R, r_1 में संबंध:—

$$R = \frac{abc}{4r_1(s-a)} \qquad r_1 = 4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$$

R, r_2 में संबंध:—

$$R = \frac{abc}{4r_2(s-b)} \qquad r_2 = 4R \cos\frac{A}{2} \sin\frac{B}{2} \cos\frac{C}{2}$$

R, r_3 में संबंध:—

$$R = \frac{abc}{4r_3(s-c)} \qquad r_3 = 4R \cos\frac{A}{2} \cos\frac{B}{2} \sin\frac{C}{2}$$

★ $r_1 + r_2 + r_3 - r = 4R$

त्रिभुज के कुछ महत्वपूर्ण गुण:-

(1.)

*⇒ त्रिभुज के कुछ महत्वपूर्ण गुणधर्म :-

(1) यदि $\cos A + \cos B + \cos C = \cancel{\sqrt{3}}\ 3/2$
तो त्रिभुज समबाहु त्रिभुज होगा।

(2) यदि $\cot A + \cot B + \cot C = \sqrt{3}$,
तो वह त्रिभुज समबाहु त्रिभुज होगा।

(3) $\sin^2 A + \sin^2 B + \sin^2 C = 2$
तो वह त्रिभुज ~~समबाहु~~ समकोण त्रिभुज होगा।

(4) यदि $\dfrac{a}{\cos A} = \dfrac{b}{\cos B} = \dfrac{c}{\cos C}$
तो त्रिभुज समबाहु Δ होगा।

(5) यदि किसी त्रिभुज में :- $a \sin A = b \sin B$
तो त्रिभुज समद्विबाहु होगा।

(6) यदि $a \cos A = b \cos B$ तो वह त्रिभुज समद्विबाहु

(7) यदि किसी Δ के लिए $\cot A \cdot \cot B \cot C > 0$
तो वह Δ, न्यूनकोण त्रिभुज होगा।

(8) यदि $\cos^2 A + \cos^2 B + \cos^2 C = 1$ तो Δ, समकोण Δ होगा।

$$\cos^2 A + \cos(B+C)\cos(B-C) = 0$$

$$\cos A = 0$$

(9) यदि किसी Δ के लिए $\dfrac{a^2-b^2}{a^2+b^2} = \dfrac{\sin(A-B)}{\sin(A+B)}$ तो

Δ या तो समकोण Δ होगा या समद्विबाहु Δ होगा।

(10) यदि किसी Δ के लिए $8R^2 = a^2 + b^2 + c^2$ तो वह Δ समकोण Δ होगा।

(11) यदि किसी Δ के लिए

$$a^2 \sin(B-C) + b^2 \sin(C-A) + c^2 \sin(A-B) = 0$$

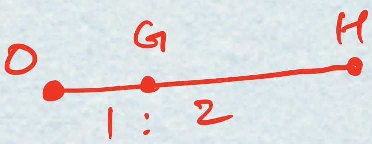
तो Δ, समद्विबाहु Δ होगा।

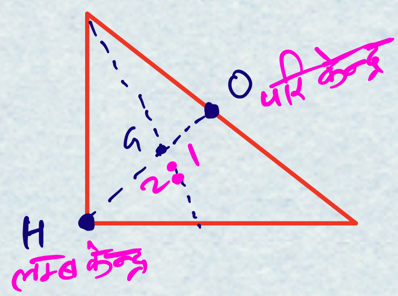
(12) समद्विबाहु Δ में- अंतः केन्द्र, परिकेन्द्र, केन्द्रक तथा लम्ब केन्द्र चारों, संरेख होते हैं।

(13) समबाहु Δ में – अंतः केन्द्र, परिकेन्द्र, केन्द्रक, लम्ब केन्द्र चारों, संपाती होते हैं।

(14) प्रत्येक Δ में परिकेन्द्र, केन्द्रक तथा लम्ब केन्द्र सदैव संरेख होते हैं

(15) केन्द्रक (G), परिकेन्द्र (O) तथा लम्बकेन्द्र (H) को 1:2 में अंतः विभाजित करता है





(16) लम्ब केन्द्र; परिकेन्द्र तथा केन्द्रक को 3:2 में बाह्य विभाजित करता है

(17) लम्ब केन्द्र तथा केन्द्रक को, परिकेन्द्र 3:1 में बाह्य विभाजित करता है

(i) $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$

(ii) $\cos(\cos^{-1} x) = x$, for all $x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x$ for all $x \in R$

(iv) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec(\sec^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot(\cot^{-1} x) = x$, for all $x \in R$.

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta & , \text{ if } \theta \in [-3\pi/2\,, -\pi/2] \\ \theta & , \text{ if } \theta \in [-\pi/2\,, \pi/2] \\ \pi-\theta & , \text{ if } \theta \in [\pi/2\,, 3\pi/2] \\ -2\pi+\theta & , \text{ if } \theta \in [3\pi/2\,, 5\pi/2] \end{cases} \quad \textit{and so on.}$$

Similarly,

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \text{ if } \theta \in [-\pi\,, 0] \\ \theta & , \text{ if } \theta \in [0, \pi] \\ 2\pi-\theta & , \text{ if } \theta \in [\pi\,, 2\pi] \\ -2\pi+\theta & , \text{ if } \theta \in [2\pi\,, 3\pi] \end{cases} \quad \textit{and so on.}$$

$$\tan^{-1}(\tan\theta) = \begin{cases} -\pi-\theta & , \text{ if } \theta \in [-3\pi/2\,, -\pi/2] \\ \theta & , \text{ if } \theta \in [-\pi/2\,, \pi/2] \\ \theta-\pi & , \text{ if } \theta \in [\pi/2\,, 3\pi/2] \\ \theta-2\pi & , \text{ if } \theta \in [3\pi/2\,, 5\pi/2] \end{cases} \quad \textit{and so on.}$$

OPERTY III (i) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, for all $x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$, for all $x \in R$

(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty]$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, for all $x \in R$

IV. 1 $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right), -1 \le x \le 1$

$\text{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right), x \le -1 \text{ or } x \ge 1$

2. $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right), -1 \le x \le 1$

$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right), x \le -1 \text{ or } x \ge 1$

3. $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right), x > 0$

$\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right) - \pi, x < 0$

$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right), x > 0$

$\cot^{-1}x = \pi + \tan^{-1}\left(\frac{1}{x}\right), x < 0$

III. 1. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $-1 \leq x \leq 1$

2. $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ for $x \in R$

3. $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$ for $x \leq -1$ or, $x \geq 1$

6.	• $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & , \text{ if } xy<1 \\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & , \text{ if } x>0,\ y>0 \text{ and } xy>1 \\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & , \text{ if } x<0,\ y<0 \text{ and } xy>1 \end{cases}$
	• $\tan^{-1}x - \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & , \text{ if } xy>-1 \\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & , \text{ if } x>0,\ y<0 \text{ and } xy<-1 \\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & , \text{ if } x<0,\ y>0 \text{ and } xy<-1 \end{cases}$
7.	• $\sin^{-1}x+\sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy<0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & , \text{ if } 0<x,\ y\le 1 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & , \text{ if } -1\le x,\ y<0 \text{ and } x^2+y^2>1 \end{cases}$
	• $\sin^{-1}x-\sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy>0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & , \text{ if } 0<x\le 1,\ -1\le y\le 0 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & , \text{ if } -1\le x<0,\ 0<y\le 1 \text{ and } x^2+y^2>1 \end{cases}$
8.	• $\cos^{-1}x+\cos^{-1}y = \begin{cases} \cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x+y\ge 0 \\ 2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x+y\le 0 \end{cases}$
	• $\cos^{-1}x-\cos^{-1}y = \begin{cases} \cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x\le y \\ -\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\} & , \text{ if } -1\le y\le 0,\ 0<x\le 1 \text{ and } x\ge y \end{cases}$

9.	$2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}) & , \text{ if } -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \frac{1}{\sqrt{2}}\le x\le 1\\ -\pi-\sin(2x\sqrt{1-x^2}), & \text{if } -1\le x\le-\frac{1}{\sqrt{2}}\end{cases}$
	$3\sin^{-1}x=\begin{cases}\sin^{-1}(3x-4x^3) & , \text{ if } -\frac{1}{2}\le x\le\frac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3) & , \text{ if } \frac{1}{2}< x\le 1\\ -\pi-\sin^{-1}(3x-4x^3), & \text{if } -1\le x<-\frac{1}{2}\end{cases}$
10.	$2\cos^{-1}x=\begin{cases}\cos^{-1}(2x^2-1) & , \text{ if } 0\le x\le 1\\ 2\pi-\cos^{-1}(2x^2-1), & \text{if } -1\le x\le 0\end{cases}$
	$3\cos^{-1}x=\begin{cases}\cos^{-1}(4x^3-3x) & , \text{ if } \frac{1}{2}\le x\le 1\\ 2\pi-\cos^{-1}(4x^3-3x), & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ 2\pi+\cos^{-1}(4x^3-3x), & \text{if } -1\le x\le-\frac{1}{2}\end{cases}$
11.	$2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right) & , \text{ if } -1<x<1\\ \pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right) & , \text{ if } x>1\\ -\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x<-1\end{cases}$
	$3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) & , \text{ if } -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) & , \text{ if } x>\frac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x<-\frac{1}{\sqrt{3}}\end{cases}$

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PROPERTY XII

(i) $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & , \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & , \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & , \text{if } x < -1 \end{cases}$$

(ii) $$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & , \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & , \text{if } -\infty < x \le 0 \end{cases}$$